

# Theoretical tools for Quantum-enhanced metrology

*the illusion of the Heisenberg scaling*

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EUROPEAN UNION  
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DEVELOPMENT FUND



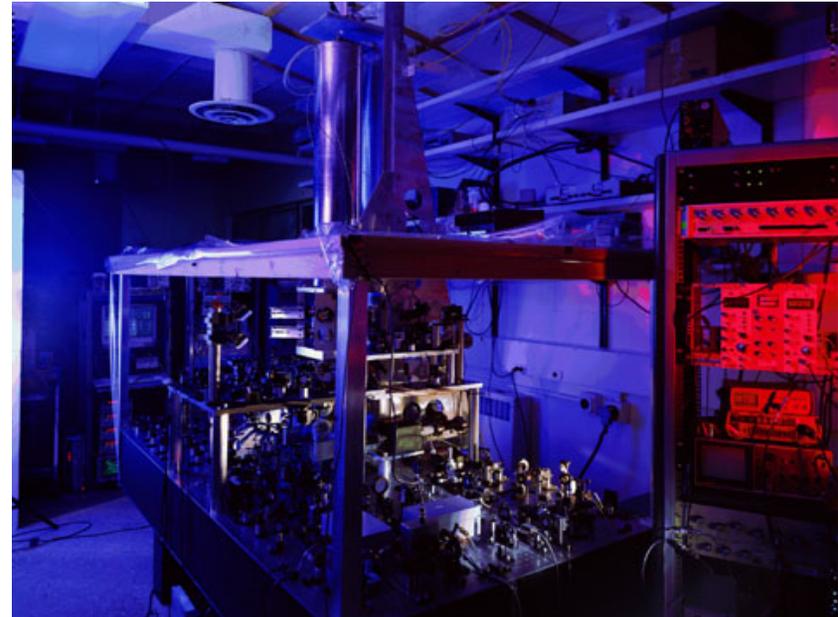
# Interferometry at its (classical) limits

LIGO - gravitational wave detector



Michelson interferometer  
 $\Delta L/L \approx 10^{-22}$

NIST - Cs fountain atomic clock



Ramsey interferometry  
 $\Delta t/t \approx 10^{-16}$

Precision limited by:

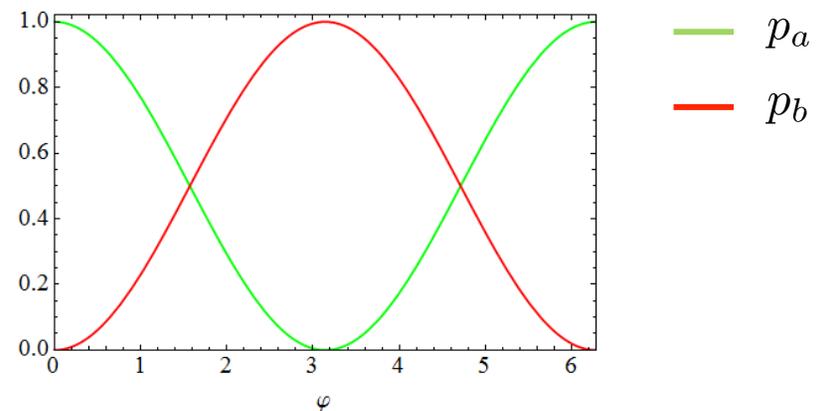
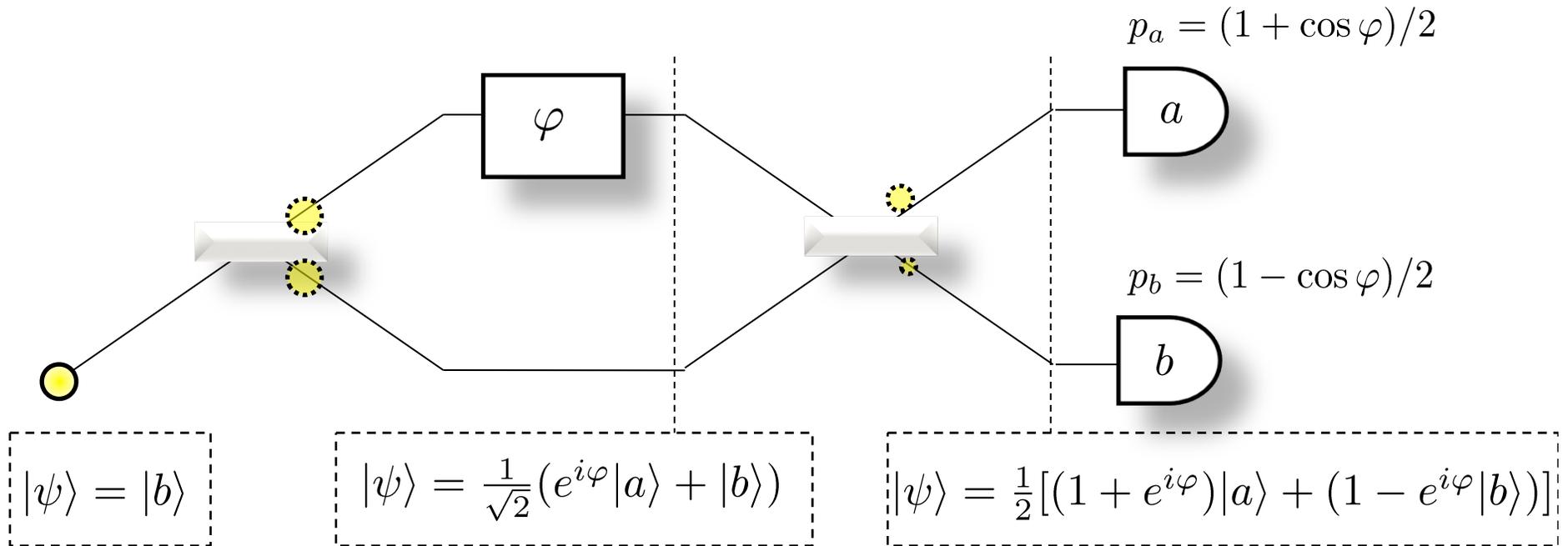
shot noise  $\propto 1/\sqrt{N}$

$N$  - number of photons

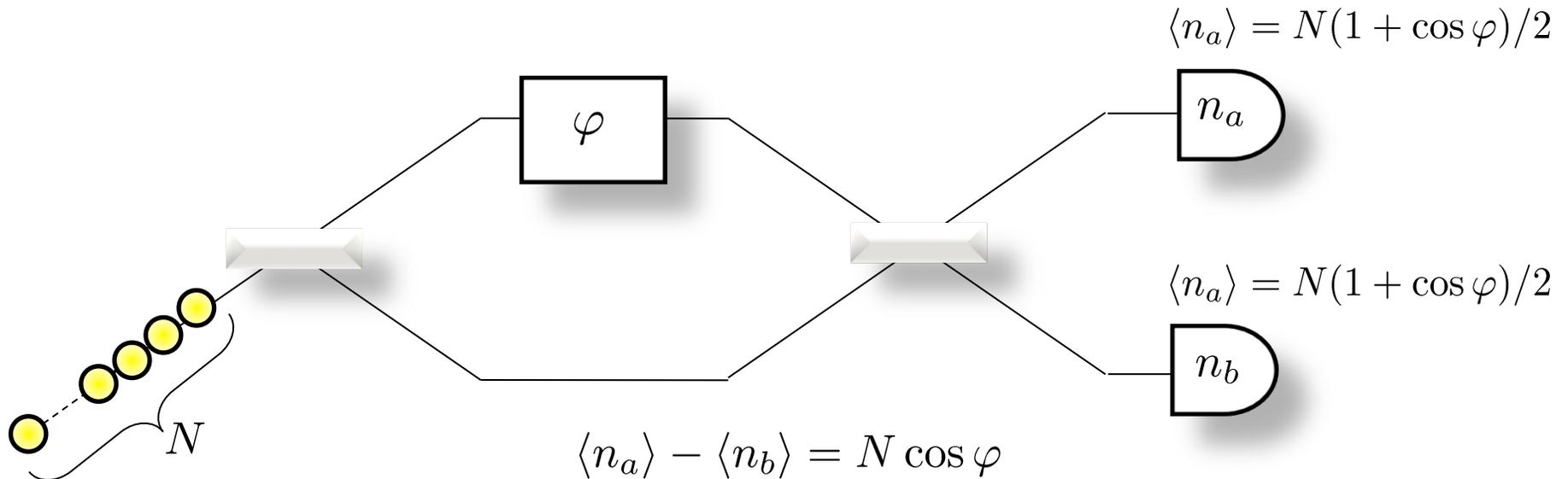
projection noise  $\propto 1/\sqrt{N}$

$N$  - number of atoms

# 1 photon in an interferometer



# $N$ independent photons



the best estimator:  $\varphi(n_a, n_b) = \arccos\left(\frac{n_a - n_b}{N}\right)$

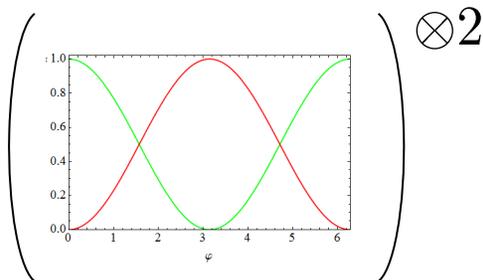
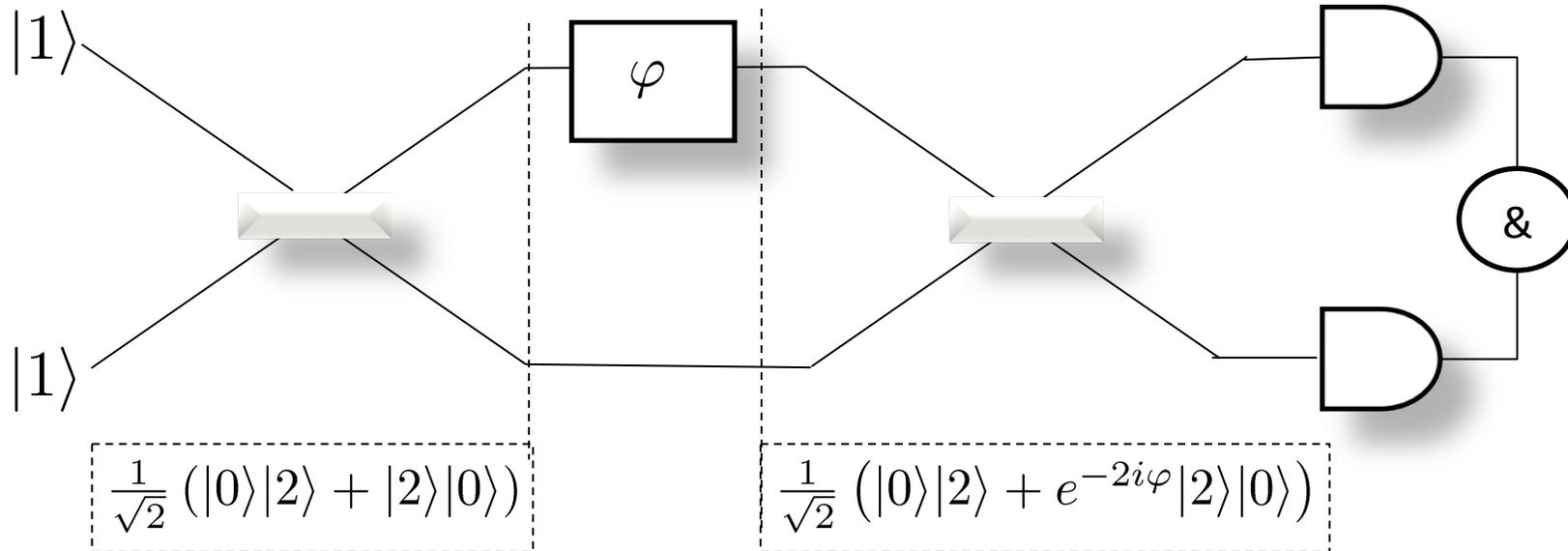
Estimator uncertainty:  $\Delta\varphi = \frac{1}{\sqrt{N}}$

Standard Quantum Limit (Shot noise limit)

The same (or worse) result for classical states of light

# Entanglement enhanced precision

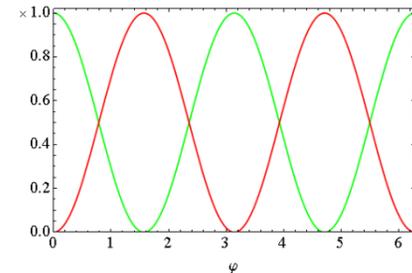
Hong-Ou-Mandel interference



$$\Delta\varphi \propto \frac{1}{\sqrt{2}}$$

$$p_{11} = \frac{1}{2}(1 + \cos 2\phi)$$

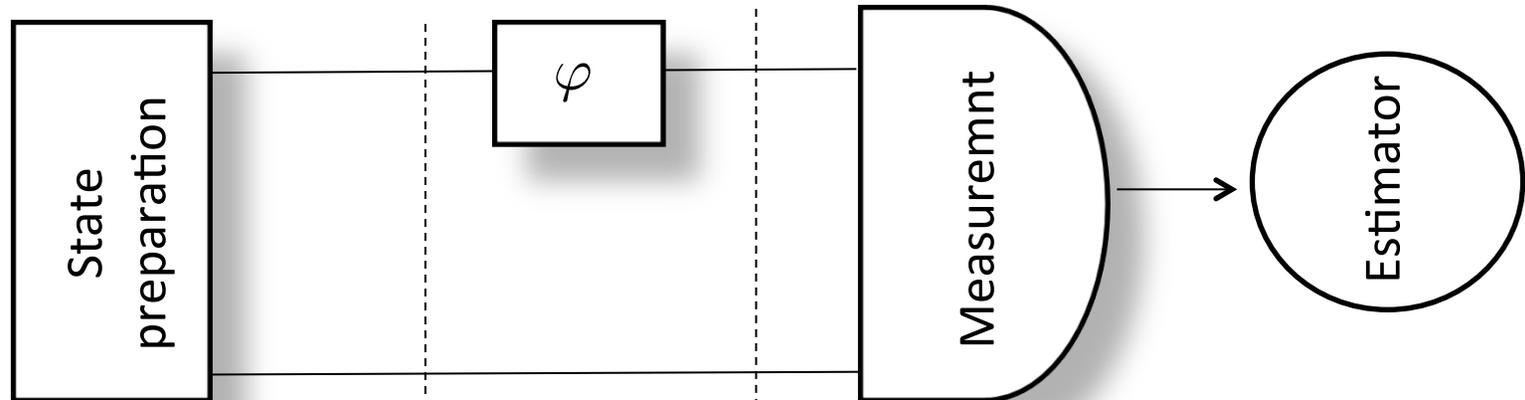
$$p_{02} + p_{20} = \frac{1}{2}(1 - \cos 2\phi)$$



$$\Delta\varphi \propto \frac{1}{2}$$

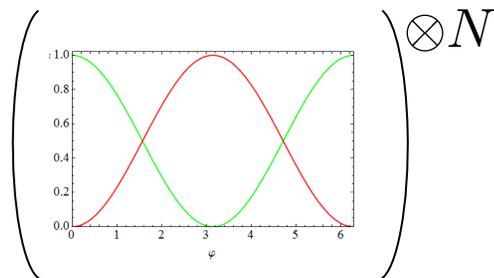
# Entanglement enhanced precision

NOON states



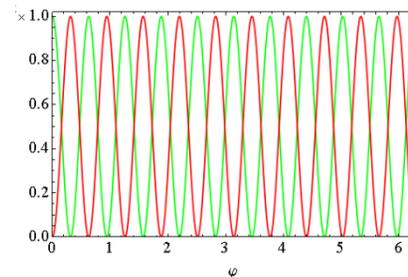
$$\frac{1}{\sqrt{2}} (|0\rangle|N\rangle + |N\rangle|0\rangle)$$

$$\frac{1}{\sqrt{2}} (|0\rangle|N\rangle + e^{-in\varphi} |N\rangle|0\rangle)$$



$$\Delta\varphi \propto \frac{1}{\sqrt{N}}$$

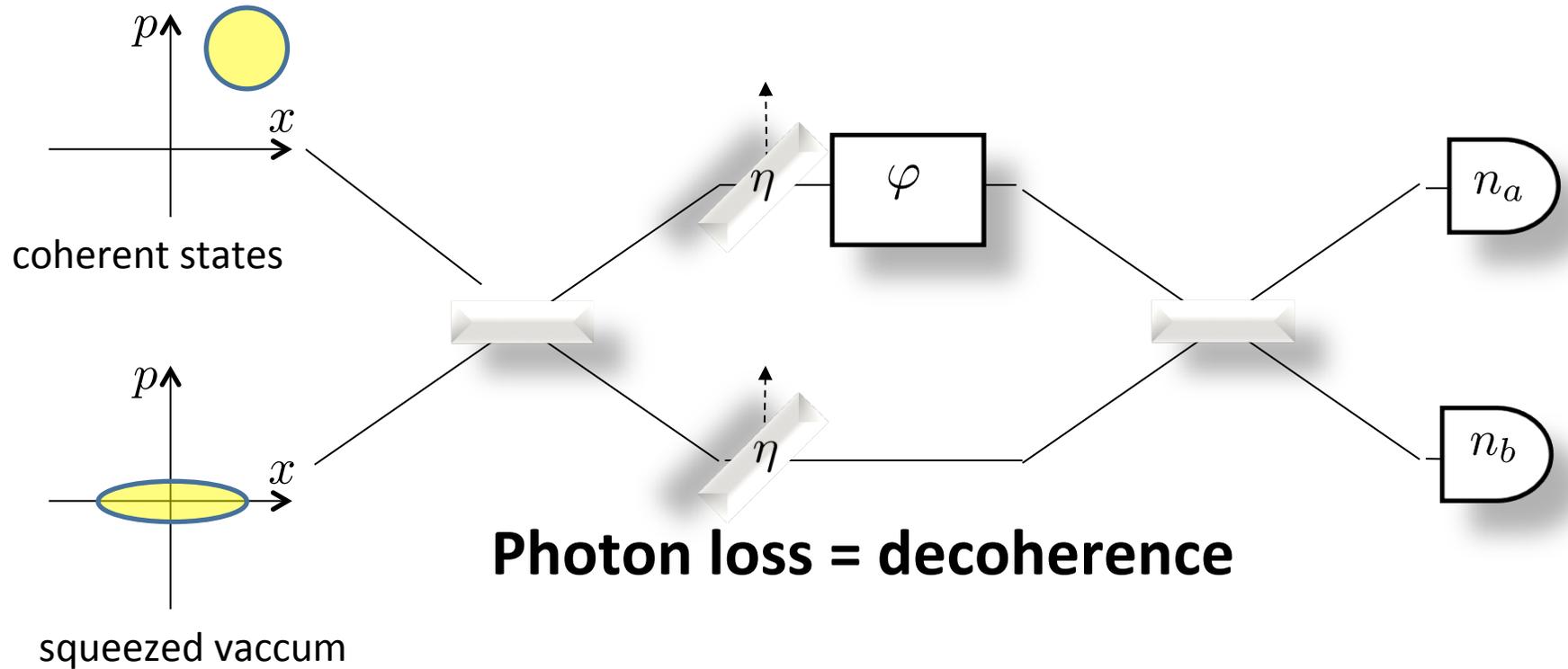
Standard Quantum Limit



$$\Delta\varphi \propto \frac{1}{N}$$

Heisenberg limit

# In practice: squeezed states



One quadrature fluctuations below vacuum fluctuations



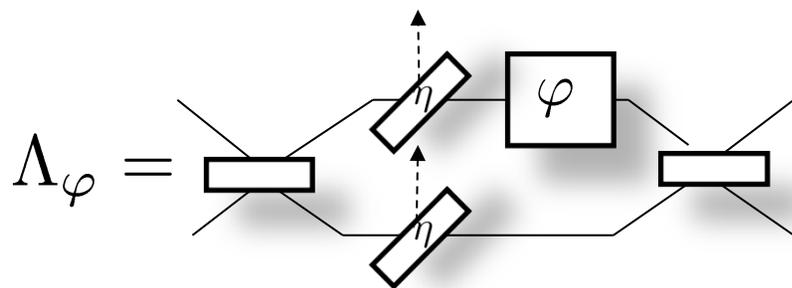
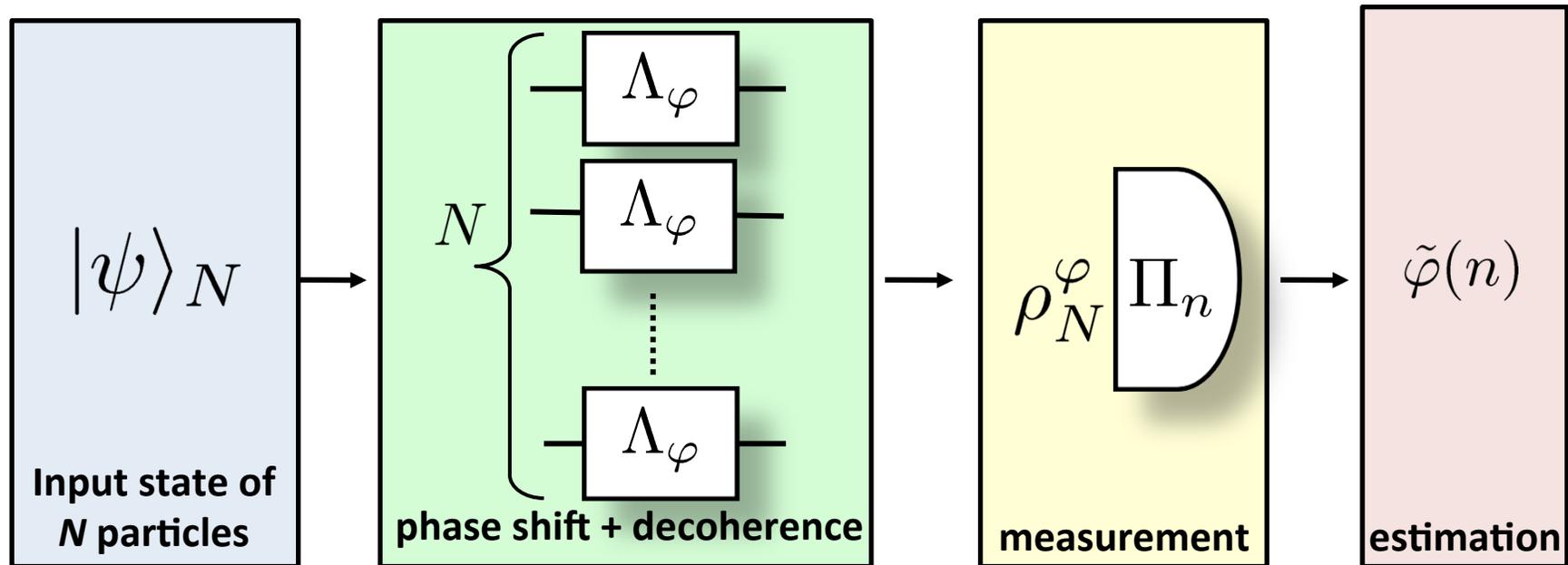
**A gravitational wave observatory operating beyond the quantum shot-noise limit**

The LIGO Scientific Collaboration <sup>††</sup>

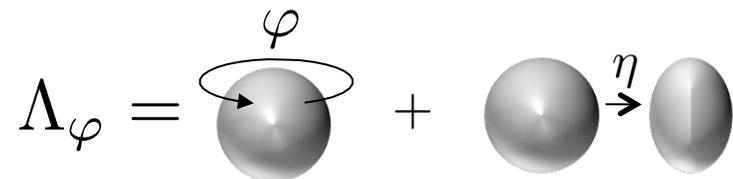
$$\frac{\Delta\varphi_{\text{squeezed}}}{\Delta\varphi_{\text{standard}}} \approx 0.66$$

**What are the fundamental bounds  
in presence of decoherence?**

# General scheme in q. metrology

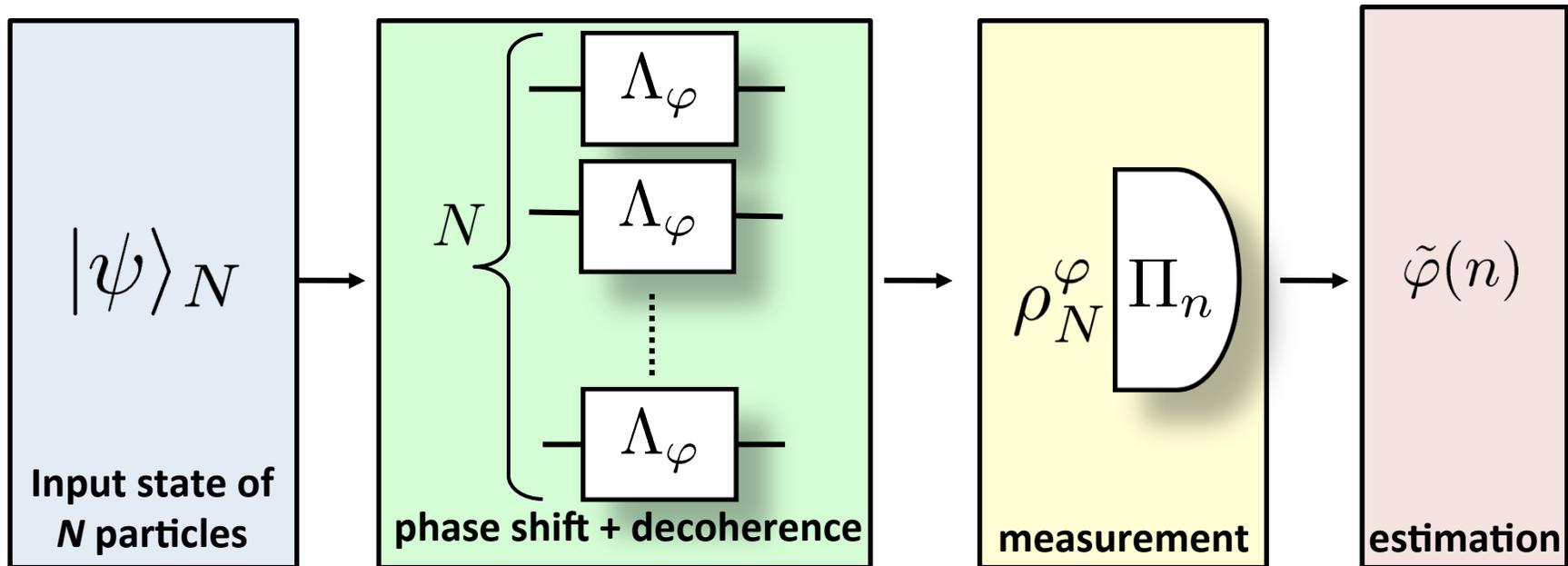


Interferometer with losses  
(gravitational wave detectors)



Qubit rotation + dephasing  
(atomic clock frequency calibrations)

# General scheme in q. metrology



Minimize  $\Delta^2\varphi$  over the choice of  $|\psi\rangle$ ,  $\Pi_n$  and  $\tilde{\varphi}$

$$\Delta^2\varphi = \langle (\tilde{\varphi} - \varphi)^2 \rangle = \int d\varphi \underbrace{p(\varphi)}_{\text{a priori knowledge}} \sum_n \underbrace{p(n|\varphi)}_{\text{Tr}(\Pi_n \rho_N^\varphi)} [\tilde{\varphi}(n) - \varphi]^2$$

$4 \sin^2 \left[ \frac{\tilde{\varphi}(n) - \varphi}{2} \right]$

Very hard problem!

$$\Delta^2 \varphi = \int d\varphi p(\varphi) \sum_n \langle \psi_\varphi | \Pi_n | \psi_\varphi \rangle [\tilde{\varphi}(n) - \varphi]^2$$

## Local approach

we want to sense small fluctuations around a known phase

$$p(\varphi) \approx \delta(\varphi - \varphi_0)$$

**Tool:** Fisher Information, Cramer-Rao bound

$$\Delta \tilde{\varphi} \geq \frac{1}{\sqrt{F}}$$

$$F = 4[\langle \psi_\varphi | \hat{n}_1^2 | \psi_\varphi \rangle - \langle \psi_\varphi | \hat{n}_1 | \psi_\varphi \rangle^2]$$

The optimal N photon state for interferometry:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|N, 0\rangle + |0, N\rangle)$$

$$\Delta \tilde{\varphi} \approx \frac{1}{N}$$

J. J. . Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, *Phys. Rev. A* **54**, R4649 (1996).

## Global approach

no a priori knowledge about the phase

$$p(\varphi) \approx \frac{1}{2\pi}$$

**Tool:** Symmetry implies a simple structure of the optimal measurement

Optimal state:  $|\psi\rangle = \sum_{n=0}^N \alpha_n |n, N-n\rangle$

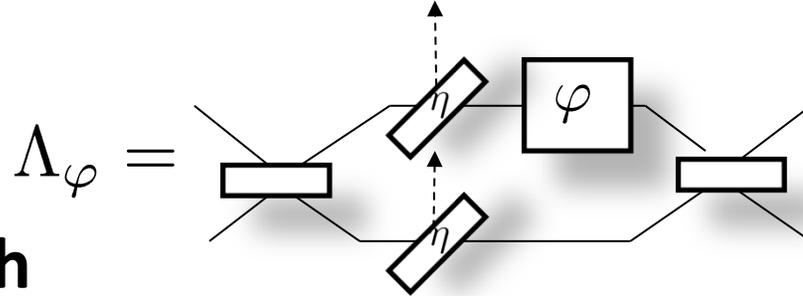
$$\alpha_n = \sqrt{\frac{2}{N+2}} \sin \left[ \frac{(n+1)\pi}{N+2} \right]$$

$$\Delta \tilde{\varphi} \approx \frac{\pi}{N+2}$$

D. W. Berry and H. M. Wiseman, *Phys. Rev. Lett.* **85**, 5098 (2000).

## Heisenberg scaling

# Impact of decoherence?



## Local approach

**Tool:** Fisher Information, Cramer-Rao bound

$$\Delta\tilde{\varphi} \geq \frac{1}{\sqrt{F}}$$



- Fisher Information calculate
- Optimal state structure

RDD, et al. PRA **80**, 013825 (2009)

U. Dorner, et al., PRL. **102**, 040403 (2009)



**Analytical lower bound:**

$$\Delta\tilde{\varphi} \geq \sqrt{\frac{1-\eta}{\eta}} \frac{1}{\sqrt{N}}$$

S. Knysh, V. Smelyanskiy, G. Durkin PRA **83**, (2011)

## Global approach

**Tool:** Symmetry implies a simple structure of the optimal measurement



- nontrivial eigenvalue problem



**Analytical lower bound:**

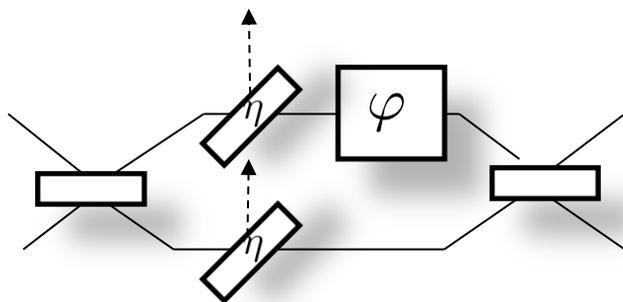
$$\Delta\tilde{\varphi} \geq \sqrt{\frac{1-\eta}{\eta}} \frac{1}{\sqrt{N}}$$

J. Kolodynski, RDD, PRA **82**,053804 (2010)

**Heisenberg scaling lost!**

the optimal

# Fundamental bound on quantum enhancement of precision



$$\Delta\tilde{\varphi}_{\text{quantum}} \geq \sqrt{\frac{1-\eta}{\eta}} \frac{1}{\sqrt{N}}$$

$$\Delta\tilde{\varphi}_{\text{classical}} = \frac{1}{\sqrt{\eta N}}$$

$$\frac{\Delta\varphi_{\text{quantum}}}{\Delta\tilde{\varphi}_{\text{classical}}} \geq \sqrt{1-\eta}$$

LETTERS

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nature  
physics

$$\eta = 0.62$$

**A gravitational wave observatory operating beyond the quantum shot-noise limit**

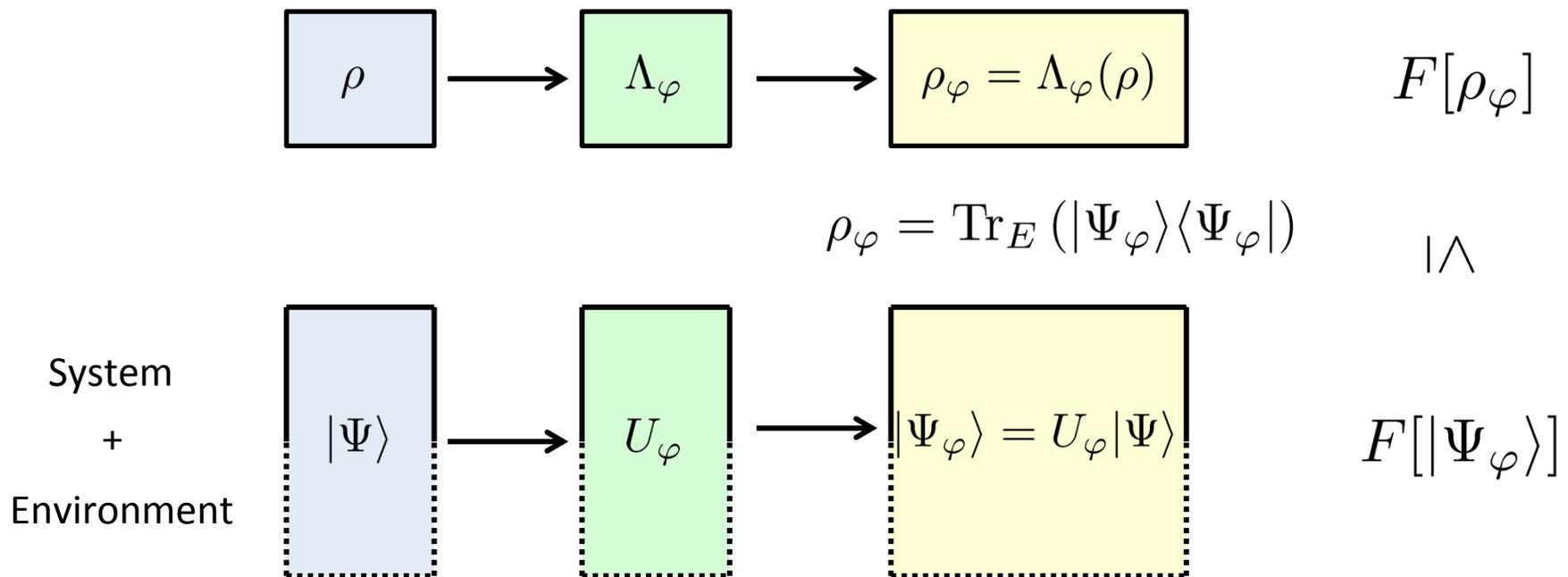
The LIGO Scientific Collaboration \*\*

$$\frac{\Delta\varphi_{\text{squeezed}}}{\Delta\varphi_{\text{coherent}}} \approx 0.66$$

$$\frac{\Delta\varphi_{\text{quantum}}}{\Delta\tilde{\varphi}_{\text{classical}}} \geq 0.617$$

# General method for other decoherence models?

## Fisher information via purifications

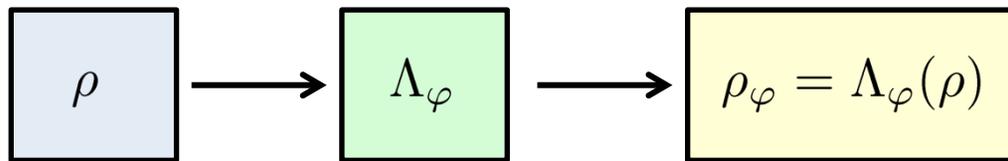


$$F[\rho_\varphi] = \min_{|\Psi_\varphi\rangle} F[|\Psi_\varphi\rangle]$$

B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Nature Physics, **7**, 406 (2011)  
 A. Fujiwara and H. Imai, J. Phys. A: Math. Theor., 41, 255304 (2008).

# General method for other decoherence models?

## Fisher information via purifications



$$F[\rho_\varphi] = \min_{|\Psi_\varphi\rangle} F[|\Psi_\varphi\rangle]$$

$$\rho_\varphi = \text{Tr}_E (|\Psi_\varphi\rangle\langle\Psi_\varphi|)$$

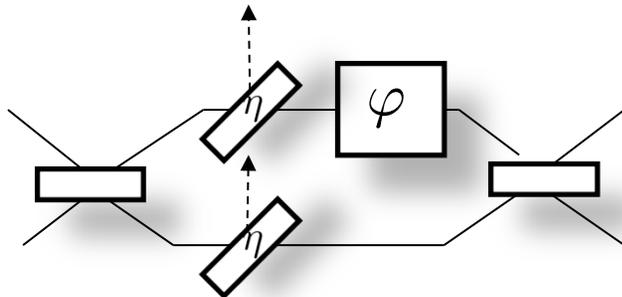
$$\Lambda_\varphi(\rho) = \sum_i K_{i,\varphi} \rho K_{i,\varphi}^\dagger$$

Kraus representation

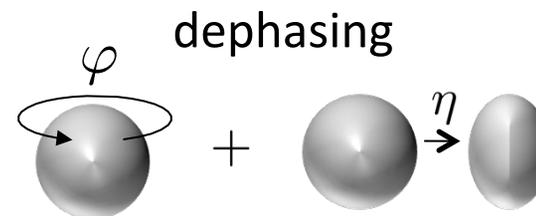
$$\tilde{K}_{i,\varphi} = \sum_j v_{ij}(\varphi) K_{j,\varphi}$$

Equivalent Kraus set

## Minimization over different Kraus representation non-trivial



$$\Delta\tilde{\varphi} \geq \sqrt{\frac{1-\eta}{\eta}} \frac{1}{\sqrt{N}}$$



$$\Delta\tilde{\varphi} \geq \frac{\sqrt{1-\eta^2}}{\eta} \frac{1}{\sqrt{N}}$$

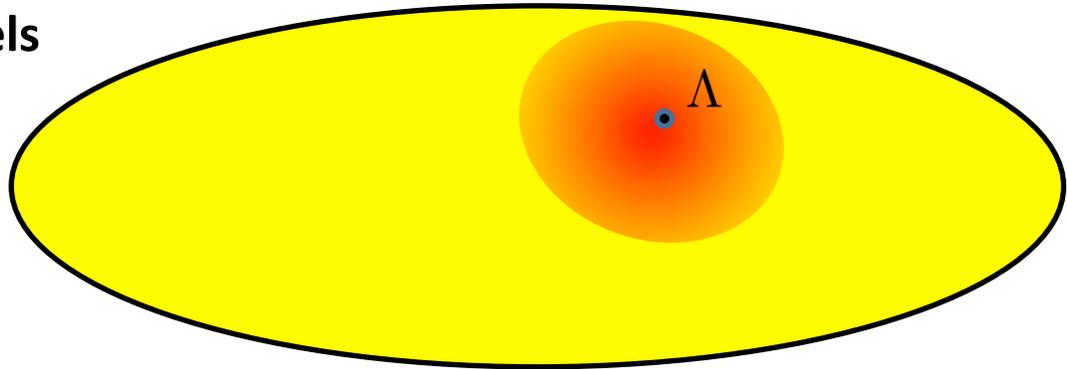
**Can you do it simpler, more  
general, more intuitive?**

**Yes!!!**

# Classical simulation of a quantum channel

Convex set of quantum channels

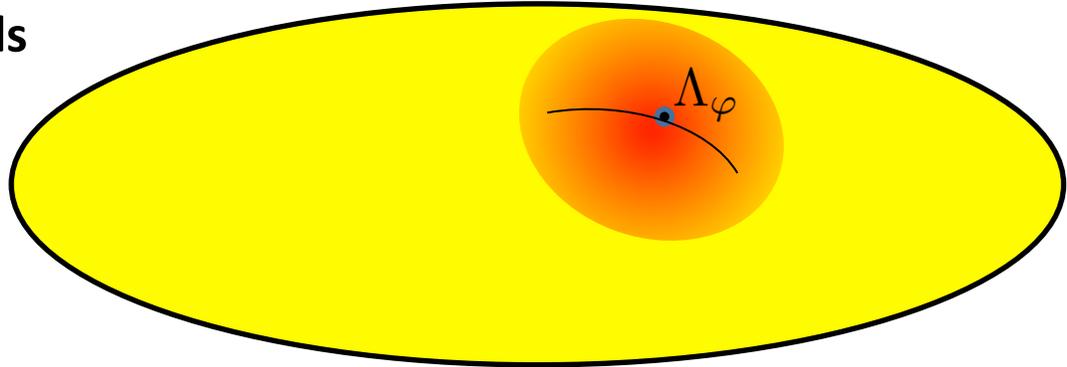
$$\Lambda = \int dX p(X) \Lambda_X$$



# Classical simulation of a quantum channel

Convex set of quantum channels

$$\Lambda_\varphi = \int dX p_\varphi(X) \Lambda_X$$



Parameter dependence moved to mixing probabilities

Before:

$$\varphi \rightarrow \Lambda_\varphi[\rho] \rightarrow \tilde{\varphi}$$

After:

$$\varphi \rightarrow p_\varphi \rightarrow X \rightarrow \Lambda_X[\rho] \rightarrow \tilde{\varphi}$$

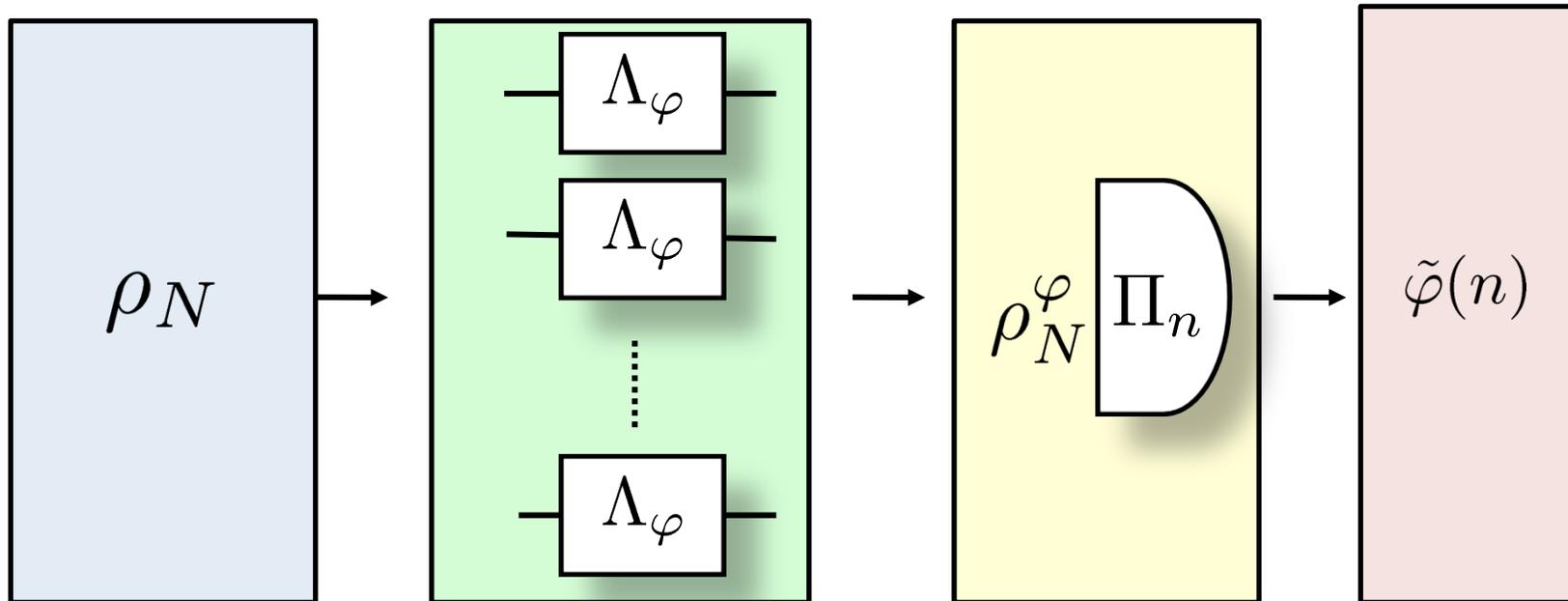
By Markov property....

Estimating  $\varphi$  directly from  $X$  is no worse than from measurement on  $\Lambda_\varphi[\rho]$

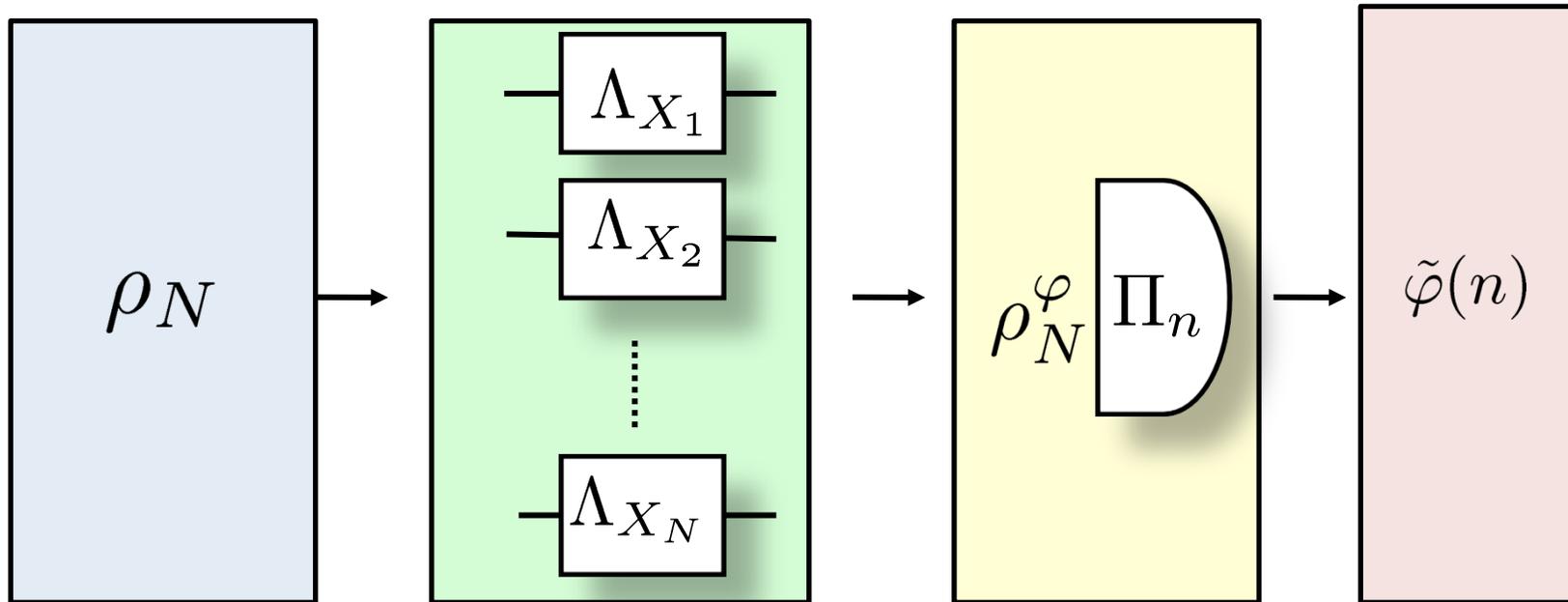
$$F_Q[\Lambda_\varphi(\rho)] \leq F_{cl}[p_\varphi(X)]$$

$$F_{cl}[p_\varphi(X)] = \int dX \frac{[\partial_\varphi p_\varphi(X)]^2}{p_\varphi(X)}$$

# Classical simulation of $N$ channels used in parallel

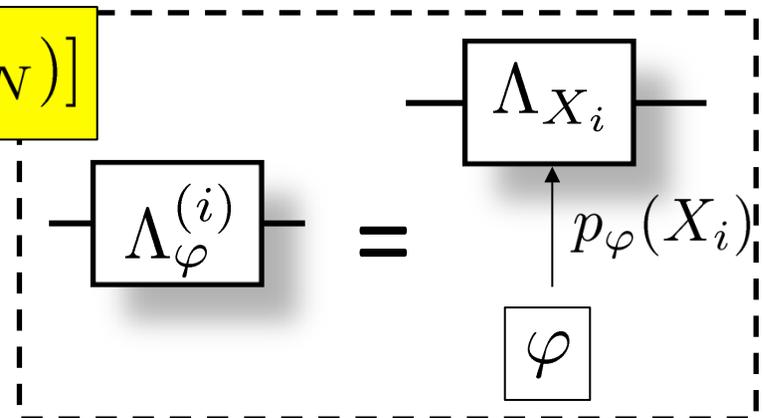


# Classical simulation of $N$ channels used in parallel

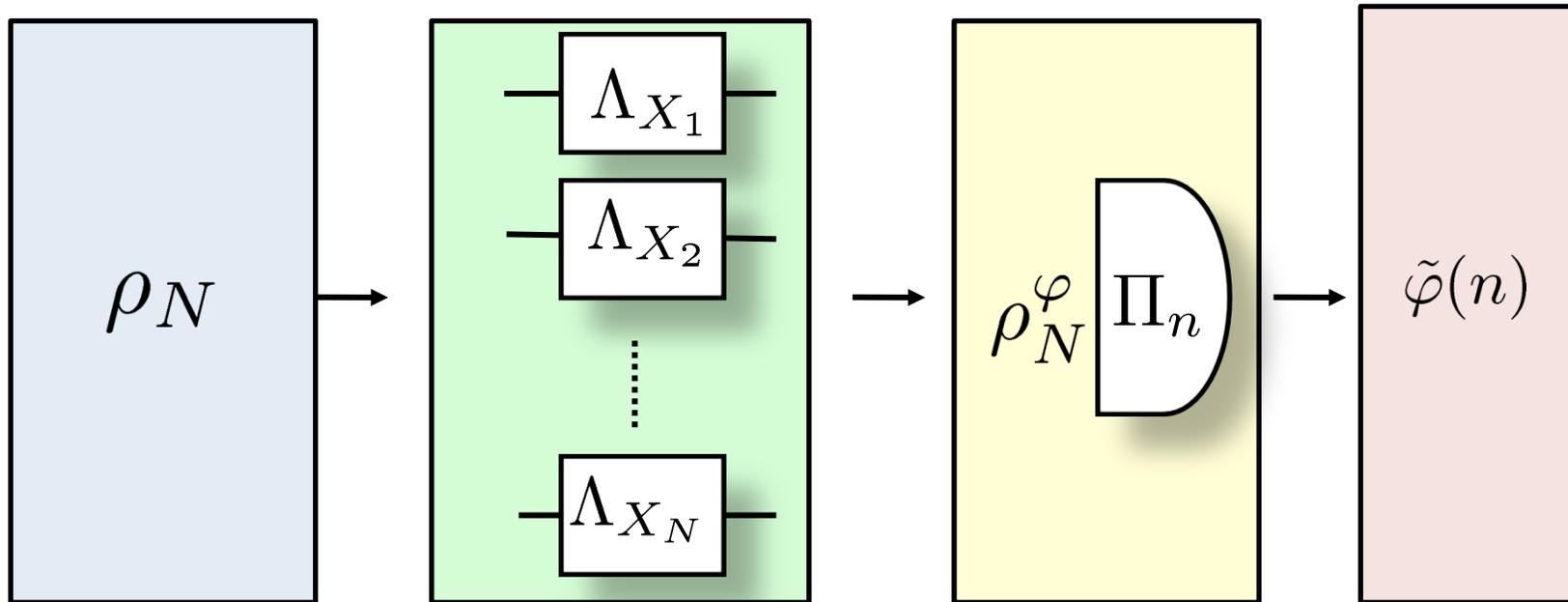


$$F_Q [\Lambda_\varphi^{\otimes N} (\rho^N)] \leq F_{cl} [p_\varphi(X_1, \dots, X_N)]$$

$X_i$  - are independent variables!



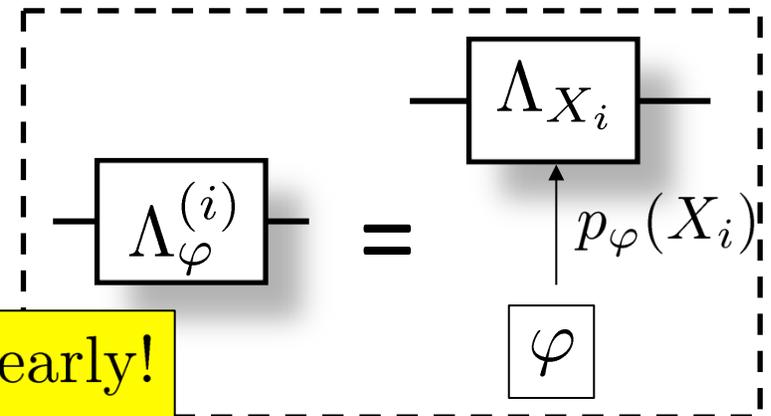
# Classical simulation of $N$ channels used in parallel



$$F_Q [\Lambda_\varphi^{\otimes N}(\rho^N)] \leq N F_{c1} [p_\varphi(X)]$$

$X_i$  - are independent variables!

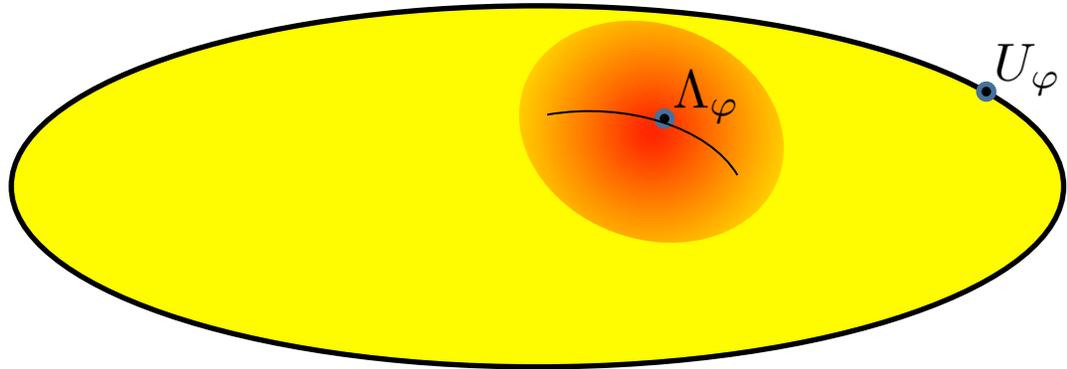
If  $F_{c1}$  is finite  $F_Q$  scales at most linearly!



# Precision bounds thanks to classical simulation

$$\Lambda_\varphi = \int dX p_\varphi(X) \Lambda_X$$

$$\Delta\varphi \geq \frac{1}{\sqrt{F_{\text{cl}}(p_\varphi)N}}$$

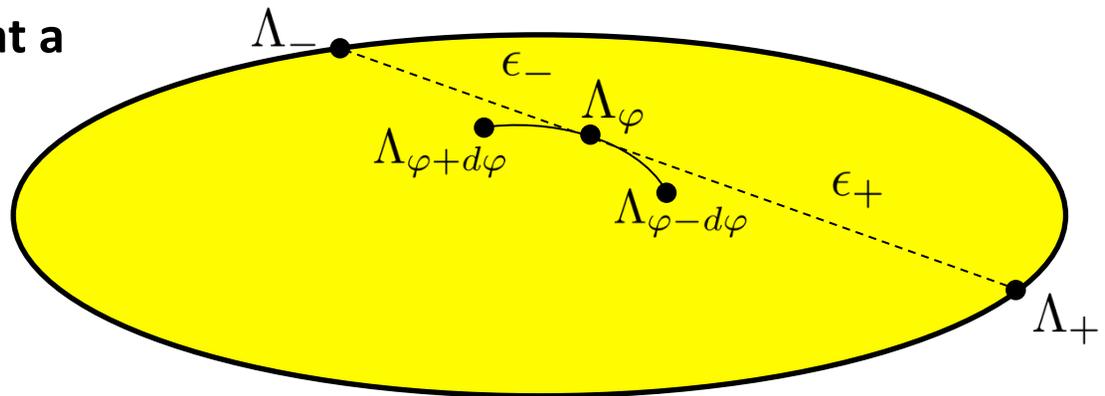


- For unitary channels  $F_{\text{cl}} = \infty$  Heisenberg scaling possible
- Generic decoherence model will manifest shot noise scaling
- To get the tightest bound we need to find the „worst” classical simulation

# The „Worst” classical simulation

Quantum Fisher Information at a given  $\varphi$  depends only on

$$\Lambda_\varphi \quad \partial_\varphi \Lambda_\varphi$$



It is enough to analyze „local classical simulation”:

$$\Lambda_\varphi = \int dX p_\varphi(X) \Lambda_X + O(d\varphi^2)$$

The „worst” classical simulation:

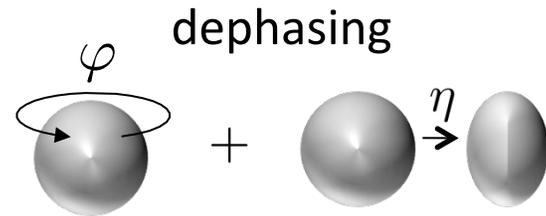
$$\Lambda_\varphi = p_+(\varphi) \Lambda_+ + p_-(\varphi) \Lambda_- + O(d\varphi^2)$$

$$\Lambda_\pm = \Lambda_\varphi \pm \frac{d\Lambda_\varphi}{d\varphi} \epsilon_\pm$$

$$\Delta\varphi \geq \sqrt{\frac{\epsilon_1 \epsilon_2}{N}}$$

Works for  $\varphi$  non-extremal channels

# Dephasing: derivation of the bound in 60 seconds!



$$\Lambda_\varphi(\rho) = U_\varphi \left( \sum_i K_i \rho K_i^\dagger \right) U_\varphi^\dagger$$

$$K_1 = \sqrt{\frac{1+\eta}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$K_2 = \sqrt{\frac{1-\eta}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

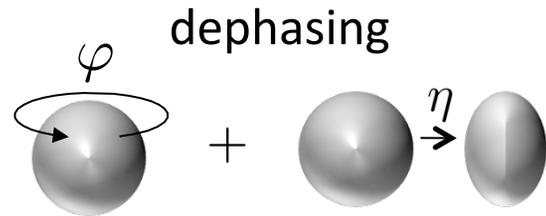
**Choi-Jamiolowski-isomorphism** (positive operators correspond to physical maps)

$$P_{\Lambda_\varphi} = \Lambda_\varphi \otimes \mathbb{1}(|\Phi\rangle\langle\Phi|) \quad |\Phi\rangle = \sum_i |i\rangle \otimes |i\rangle \quad \text{we look for } \varepsilon_\pm \text{ such that}$$

$$P_{\Lambda_\varphi} \pm \varepsilon_\pm \partial_\varphi P_{\Lambda_\varphi} \geq 0$$

$$P_{\Lambda_\varphi} = \begin{pmatrix} 1 & 0 & 0 & e^{i\varphi\eta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-i\varphi\eta} & 0 & 0 & 1 \end{pmatrix}$$

# Dephasing: derivation of the bound in 60 seconds!



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**Choi-Jamiolowski-isomorphism** (positive operators correspond to physical maps)

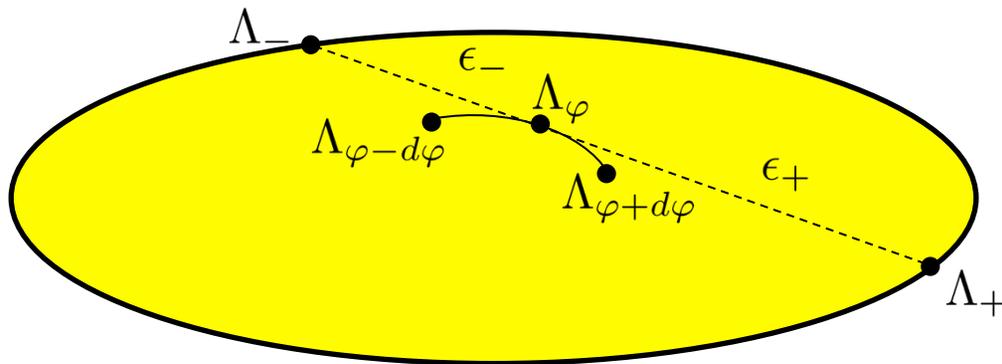
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$$P_{\Lambda_\varphi} + \varepsilon \partial_\varphi P_{\Lambda_\varphi} = \begin{pmatrix} 1 & 0 & 0 & e^{i\varphi}\eta(1+i\varepsilon) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-i\varphi}\eta(1-i\varepsilon) & 0 & 0 & 1 \end{pmatrix} \quad \begin{aligned} P_{\Lambda_\varphi} \pm \varepsilon_\pm \partial_\varphi P_{\Lambda_\varphi} &\geq 0 \\ \eta^2(1+\varepsilon^2) &\leq 1 \\ \varepsilon &\leq \frac{\sqrt{1-\eta^2}}{\eta} \end{aligned}$$

$$\Delta\tilde{\varphi} \geq \sqrt{\frac{\varepsilon_+ \varepsilon_-}{N}} = \frac{\sqrt{1-\eta^2}}{\eta} \frac{1}{\sqrt{N}}$$

# Summary

- Heisenberg scaling is lost for a generic decoherence channel even for infinitesimal noise
- Simple bounds on precision can be derived using classical simulation idea
- Channels for which classical simulation does not work ( $\varphi$  extremal channels) have less Kraus operators, other methods easier to apply



$$\Delta\varphi \geq \sqrt{\frac{\epsilon_1 \epsilon_2}{N}}$$